Parameter-Robust Domain Decomposition Methods for Semilinear Singularly Perturbed Parabolic Reaction-Diffusion Equations in a Composed Domain*

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1. Introduction

The solutions of boundary value problems in composed domains have singularities generated by discontinuities in the coefficients and right-hand side. When the problem in question is singularly perturbed (with a small parameter $\varepsilon \in (0,1]$ multiplying the highest derivatives in the differential equation), redundant singularities may appear, namely boundary and interior layers. The fact that the partial differential equation can be nonlinear complicates the solution process. Because of the thin layers, standard numerical methods applied to problems from this class yield unsatisfactorily large errors for small values of the perturbation parameter ε. Model examples show that the errors of standard finite difference methods on uniform meshes grow and become comparable with the solution for $\varepsilon \approx N^{-1}$, where N is the number of mesh intervals in x.

In a space-time domain composed of two rectangles, we consider a singularly perturbed boundary value problem for a semilinear parabolic reaction-diffusion equation. The conjugation conditions which reflect the conservation laws are given on the interface between the subdomains. As $\varepsilon \to 0$, the solution has boundary layers and a transient (interior) layer. The interior layer appears on both sides of the interface. Problems of this type arise in numerical modelling of nonstationary heat and mass transfer in a stratified composite medium when the coefficients of heat and/or mass transfer are small.

For this singularly perturbed problem in the composed domain whose solution has several singularities, our goal is to develop ε-uniformly convergent numerical schemes based on a domain decomposition method so that the problem to be solved on each subdomain has no more than a single singularity. Using standard finite difference operators on piecewise uniform meshes condensing in the layer regions, we first construct a (nonlinear) difference scheme that converges ε-uniformly at the rate $O(N^{-2}\ln^2 N + N_0^{-1})$, where N_0 is the number of mesh intervals in t. Making use of this base scheme, we construct domain decomposition schemes, in which the linearized subproblems on overlapping subdomains can be solved sequentially or in parallel (independently of each other). We give conditions under which the overlapping Schwarz-like method developed in this paper is robust in the sense that its solutions converge ε-uniformly (at the same rate as the base scheme) as the number of mesh points grow.

Note that the use of piecewise uniform fitted meshes for solving linear partial differential equations in composed domains was considered in [1]. Grid approximations to quasilinear parabolic equations in homogeneous domains (the coefficients of these equations are continuous on the solution domain) were studied, e.g., in [2]. A technique how to reduce the solution process for singularly perturbed problems with multiple singularities to the solution of a few

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local problems on subdomains each of which involves no more than one singularity, which essentially simplifies the solution of the problem of our interest, was not practically considered. A similar approach was applied in [3] where a parameter-uniform overlapping Schwarz method was constructed for linear one-dimensional reaction-diffusion problems with boundary layers in homogeneous domains (see also there a review of the bibliography).

2. Problem formulation

In a space-time domain $\overline{G}$ composed of two rectangular subdomains $\overline{G}_1$ and $\overline{G}_2$, we consider a singularly perturbed boundary value problem for the semilinear parabolic reaction-diffusion equation

$$L_k\, u(x,t) \equiv \left\{ \varepsilon^2 a_k(x,t) \frac{\partial^2}{\partial x^2} - p_k(x,t) \frac{\partial}{\partial t} \right\} u(x,t) = f_k(x,t,u(x,t)), \quad (x,t) \in G_k, \tag{2.1a}$$

$$u(x,t) = \varphi(x,t), \quad (x,t) \in S, \qquad k = 1,2. \tag{2.1b}$$

Here

$$G = D \times (0,T], \quad D = \{x:\ -d_0 < x < d^0\}, \quad d_0,\ d^0 > 0; \quad S = \overline{G} \setminus G,$$

the set $\overline{D}$ consists of two intervals $\overline{D}_1$ and $\overline{D}_2$:

$$D_1 = (-d_0, 0), \ \ D_2 = (0, d^0), \ \ \text{correspondingly,} \ \ G_k = D_k \times (0,T], \ \ k = 1,2.$$

On the interface between the subdomains, i.e., on the set $S^* = \Gamma^* \times (0,T]$, where $\Gamma^* = \{x = 0\}$, we have the conjugation condition

$$l\, u(x,t) \equiv \varepsilon \Big[a(x,t) \frac{\partial}{\partial x} u(x,t) \Big] = 0, \quad [\, u(x,t) \,] = 0, \quad (x,t) \in S^*, \tag{2.1c}$$

which responds to the continuity of the solution and of the diffusion flux when passing across the interface. The perturbation parameter ε takes arbitrary values in the half-open interval $(0,1]$. In (2.1) the functions $a_k(x,t)$, $p_k(x,t)$ and $f_k(x,t,u)$ are assumed to be sufficiently smooth on $\overline{G}_k$ and $\overline{G}_k \times R$, respectively, and $\varphi(x,t)$ is sufficiently smooth on S_j, $j = 0,1,2$, $\varphi \in C(S)$, where $S = S^L \cup S_0$, $S^L = S_1 \cup S_2$, $S_0 = \overline{S}_0$ is the lower base, S_1 and S_2 are the left and right lateral boundaries. It is assumed also that [1]

$$\begin{aligned} &0 < a_0 \le a_k(x,t) \le a^0, \quad 0 < p_0 \le p_k(x,t) \le p^0, \quad (x,t) \in \overline{G}_k; \\ &|f_k(x,t,u)| \le M, \ \ (x,t,u) \in \overline{G}_k \times R; \ \ |\varphi(x,t)| \le M, \ \ (x,t) \in S; \ \ k = 1,2. \end{aligned} \tag{2.2}$$

The symbol $[\, v \,]$ in (2.1c) denotes the jump of a function v when passing through the interface S^* from G_1 to G_2: $[\, u(x,t) \,] = \lim\limits_{x_2 \to x} u(x_2,t) - \lim\limits_{x_1 \to x} u(x_1,t)$, $\Big[a(x,t) \frac{\partial}{\partial x} u(x,t) \Big] = \lim\limits_{x_2 \to x} a_2(x_2,t) \frac{\partial}{\partial x} u(x_2,t) - \lim\limits_{x_1 \to x} a_1(x_1,t) \frac{\partial}{\partial x} u(x_1,t)$, $(x,t) \in S^*$, $(x_1,t) \in G_1$, $(x_2,t) \in G_2$.

It is convenient to write equation (2.1a) also in the form

$$L\, u(x,t) = f(x,t,u(x,t)), \quad (x,t) \in G \setminus S^*, \tag{2.1d}$$

where $L = L_k$, $f(x,t,u) = f_k(x,t,u)$, $(x,t) \in G_k$. At the corner points $S_0 \cap \overline{S}^L$, compatibility conditions are assumed to be satisfied so that the solution is sufficiently smooth for each fixed value of the parameter ε. For simplicity, we assume that

$$\frac{\partial}{\partial u} f_k(x,t,u) \ge 0, \quad \Big| \frac{\partial^2}{\partial u^2} f_k(x,t,u) \Big| \le M, \quad (x,t,u) \in \overline{G}_k \times R, \ \ k = 1,2. \tag{2.3}$$

[1] Here and below M (m) denote generic sufficiently large (small) positive constants that do not depend on ε and the discretization parameters. Throughout the paper, the notation $w_{(j.k)}$ means that the symbol w is introduced in formula $(j.k)$.

This condition ensures the existence and uniqueness of a solution for any values of ε from (0,1].

As $\varepsilon \to 0$, the solution exhibits boundary layers (in a neighbourhood of the parabolic boundary S^L) and a transient (interior) layer that occurs on both sides of the interface S^*.

3. Finite difference schemes

1. For the boundary value problem (2.1) we construct a base finite difference scheme that converges ε-uniformly, i.e., with error bounds independent of the parameter ε.

On the segment $\overline{G}$ we introduce the mesh

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \tag{3.1}$$

where $\overline{\omega}$ and $\overline{\omega}_0$ are generally nonuniform meshes on $[-d_0, d^0]$ and $[0, T_0]$, respectively; the point $x = 0$ belongs to the mesh $\overline{\omega}$. Let $h^i = x^{i+1} - x^i$, $x^i, x^{i+1} \in \overline{\omega}$, $h = \max_i h^i$; $h_t^j = t^{j+1} - t^j$, t^j, $t^{j+1} \in \overline{\omega}_0$, $h_t = \max_j h_t^j$; $N+1$ and N_0+1 are the numbers of mesh points in the meshes $\overline{\omega}$ and $\overline{\omega}_0$. We assume that $h \le MN^{-1}$, $h_t \le MN_0^{-1}$. Denote the node $x = 0$ by x^{i_0}.

The equations (2.1a), (2.1b) are approximated by the difference scheme [4]

$$\Lambda z(x,t) = f(x,t,z(x,t)),\ \ (x,t) \in G_h \setminus S^*,\ \ z(x,t) = \varphi(x,t),\ \ (x,t) \in S_h. \tag{3.2a}$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$, $\Lambda \equiv \varepsilon^2 a_k(x,t)\delta_{\overline{x}\widehat{x}} - p_k(x)\delta_{\overline{t}}$, $(x,t) \in G_{kh}$, $k = 1, 2$, $\delta_{\overline{x}\widehat{x}}\, z(x,t)$ and $\delta_x\, z(x,t)$, $\delta_{\overline{t}} z(x,t)$ are the second and first (forward and backward) difference derivatives, $\delta_{\overline{x}\widehat{x}}\, z(x,t) = 2(h^{i-1}+h^i)^{-1}(\delta_x z(x,t) - \delta_{\overline{x}} z(x,t))$, $\delta_x z(x,t) = (h^i)^{-1}(z(x^{i+1},t) - z(x,t))$, $\delta_{\overline{x}}\, z(x,t) = (h^{i-1})^{-1}(z(x,t) - z(x^{i-1},t))$, $x = x^i$, $\delta_{\overline{t}}\, z(x,t) = (h_t^{j-1})^{-1}(z(x,t) - z(x,t^{j-1}))$, $t = t^j$.

We approximate the conjugation condition (2.1c) by the discrete equation

$$\varepsilon \left\{ a_2(x,t)\, \delta_x - p^- \left(a_1(x,t)\, \delta_x \right) \right\} z(x,t) = 0, \quad (x,t) \in S_h^*, \quad x = x^{i_0}$$

where p^- is the left–shift operator such that $p^-(a_1(x^i,t)\, \delta_x z(x^i,t)) = a_1(x^{i-1},t)\, \delta_x\, z(x^{i-1},t) = a_1(x^{i-1},t)\, \delta_{\overline{x}} z(x^i,t)$. The discrete conjugation condition can be written in the "compact" form similar to (3.2a)

$$\Lambda z(x,t) \equiv \varepsilon \delta_{\widehat{x}} \left(a(x+0,t)\, \delta_x z(x,t) \right) = f(x,t,z(x,t)), \quad (x,t) \in S_h^*, \tag{3.2b}$$

where $f_{(3.2b)}(x,t,z) = 0$ for $(x,t) \in S_h^*$ (at $x = x^{i_0}$), the difference derivative $\delta_{\widehat{x}}\, z(x,t)$ is defined by [4] $\delta_{\widehat{x}}\, z(x,t) = 2(h^{i-1}+h^i)^{-1}(z(x^{i+1},t) - z(x,t))$, $x = x^i$.

The difference scheme (3.2), (3.1) is monotone [4] ε-uniformly. Using the technique of majorant functions and *a priori* estimates, similarly to [1], we find the error estimate

$$|u(x,t) - z(x,t)| \le M[\varepsilon^{-1}N^{-1} + N_0^{-1}], \qquad (x,t) \in \overline{G}_h.$$

On the uniform mesh $\overline{G}_h^u$, where $\overline{\omega} = \overline{\omega}^u$ is a uniform mesh in x with stepsize $h = (d^0 + d_0)N^{-1}$, $\overline{\omega}_0$ is a uniform mesh in time with step $h_t = T/N_0$, we obtain

$$|u(x,t) - z(x,t)| \le M[(\varepsilon + N^{-1})^{-2}N^{-2} + N_0^{-1}], \qquad (x,t) \in \overline{G}_h^u.$$

Thus, scheme (3.2) on arbitrary (in particular, uniform) meshes converges for fixed values of the parameter ε, however, it does not converge ε-uniformly.

To solve problem (2.1), we use a piecewise uniform mesh condensing in the boundary and interior layers [1]

$$\overline{G}_h = \overline{G}_h^\star = \overline{\omega}^\star(\sigma_1, \sigma_2) \times \overline{\omega}_0, \tag{3.3}$$

where $\overline{\omega}_0$ is a uniform mesh with step $h_t = T/N_0$, $\overline{\omega}^\star$ is a piecewise uniform mesh on $[-d_0, d^0]$. We construct the mesh $\overline{\omega}^\star$ by dividing $[-d_0, d^0]$ into 6 intervals $[-d_0, -d_0+\sigma_1]$, $[-d_0+\sigma_1, -\sigma_1]$, $[-\sigma_1, 0]$, $[0, \sigma_2]$, $[\sigma_2, d^0 - \sigma_2]$ and $[d^0 - \sigma_2, d^0]$. The mesh width on each subinterval is constant and equal to $h^{(1)} = 8\sigma_1 N^{-1}$ on the intervals $[-d_0, -d_0+\sigma_1]$ and $[-\sigma_1, 0]$, $h^{(2)} = 4\,(d_0 - 2\sigma_1)\, N^{-1}$

on the interval $[-d_0+\sigma_1, -\sigma_1]$, $h^{(3)} = 8\sigma_2 N^{-1}$ on the intervals $[0, \sigma_2]$ and $[d^0-\sigma_2, d^0]$, and $h^{(4)} = 4\left(d^0 - 2\sigma_2\right) N^{-1}$ on the interval $[\sigma_2, d^0-\sigma_2]$. The parameters σ_1 and σ_2 are taken to be $\sigma_1 = \min\left[4^{-1}d_0,\, M_1\,\varepsilon \ln N\right]$, $\sigma_2 = \min\left[4^{-1}d^0,\, M_2\,\varepsilon \ln N\right]$, where M_1, $M_2 > 0$ are arbitrary constants chosen according to *a priori* estimates, similarly to [1].

Theorem 1 *Let the data of the boundary value problem* (2.1) *satisfy conditions* (2.2), (2.3), *and let* $a_k, p_k \in C^{4+\alpha}(\overline{G}_k)$, $f_k \in C^{4+\alpha}(\overline{G}_k \times R)$, $\alpha > 0$, $k = 1, 2$. *Then the difference scheme* (3.2), (3.3) *converges* ε*-uniformly with error bounds*

$$|\, u(x,t) - z(x,t)\,| \le M\left[N^{-2}\ln^2 N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h^{\star}. \tag{3.4}$$

2. The above difference scheme is nonlinear. In this subsection we construct a difference scheme that is linear at each time level. To this end, we approximate a problem for the linearized equation where the unknown function in the nonlinear term is taken at the previous time level (similarly to [5]) as follows:

$$\begin{aligned} \Lambda_{(3.5)}\, z(x,t) &= f(x,t,\check{z}(x,t)), \quad (x,t) \in G_h, \\ z(x,t) &= \varphi(x,t), \quad (x,t) \in S_h. \end{aligned} \tag{3.5}$$

Here $\Lambda_{(3.5)} z(x,t) = \Lambda_{(3.2a)} z(x,t)$, $(x,t) \in G_h \setminus S^*$, $\Lambda_{(3.5)} z(x,t) = \Lambda_{(3.2b)} z(x,t)$, $(x,t) \in G_h \cap S^*$, $\check{z}(x,t) = z(x,t-h_t)$, $(x,t) \in G_h$.

For simplicity, we assume that

$$\frac{\partial}{\partial u} f(x,t,u) \le M_0, \quad (x,t,u) \in \overline{G} \times R.$$

Under the condition

$$h_t \le M_0^{-1} \min_{k,\,\overline{G}_k} p(x,t) \tag{3.6}$$

the difference scheme (3.5), (3.1) is monotone. Under condition (3.6) we obtain the estimate similar to (3.4)

$$|\, u(x,t) - z_{(3.5)}(x,t)\,| \;\le\; \left[N^{-2}\ln^2 N + N_0^{-1}\right], \quad (x,t) \in \overline{G}_h^{\star}, \tag{3.7}$$

where $z_{(3.5)}(x,t)$ is a solution of scheme (3.5) on the mesh (3.3).

Theorem 2 *Let the hypotheses of Theorem* 1 *and condition* (3.6) *be fulfilled. Then the solution of scheme* (3.5), (3.3) *for* $N \to \infty$ *converges to the solution of the boundary value problem* (2.1) ε*-uniformly. The numerical solutions satisfy estimate* (3.7).

Our primary concern here is to construct a decomposition of this base discrete problem such that the solution of the differential problem on each subdomain of the decomposition has no more than one of the singularities (either boundary layer or the interior layer).

4. Discrete overlapping Schwarz-like method

For the scheme (3.5) we describe an overlapping domain decomposition method [5–8]. Let open subdomains D^k, $k = 1, ..., K$ cover the domain D: $D = \bigcup_{k=1}^{K} D^k$, and let $G^k = D^k \times (0, T]$, $k = 1, ..., K$; $K \ge 3$. We denote the minimal overlap of the sets D^k and $D^{[k]} = \bigcup_{i=1,\, i\ne k}^{K} D^i$ by Δ^k, and the smallest value of Δ^k by Δ, i.e., $\min\limits_{j,\; x^1,\; x^2} \rho(x^1, x^2) = \Delta$, $x^1 \in \overline{D}^{\,j}$, $x^2 \in \overline{D}^{\,[j]}$, $x^1, x^2 \notin \left\{D^j \cap D^{[j]}\right\}$, $j = 1, ..., J$, where $\rho(x^1, x^2)$ is the distance between x^1, x^2. Generally speaking, $\Delta = \Delta(\varepsilon)$.

Let each D^k be partitioned into P disjoint (possibly empty) sets:

$$D^k = \bigcup_{p=1}^{P} D_p^k\,, \quad k = 1, \ldots, K, \quad \overline{D}_{p_1}^k \bigcap \overline{D}_{p_2}^k = \emptyset, \;\; p_1 \ne p_2. \tag{4.1}$$

We set $G_p^k = D_p^k \times \overline{\omega}_0$, $p = 1, ..., P$, $k = 1, ..., K$. By $G(t_1)$ we denote the strip $G(t_1) = \{(x,t) : (x,t) \in G,\ t_1 < t \le t_1 + h_t\}$, $t_1, t_1 + h_t \in \overline{\omega}_0$, $\overline{\omega}_0 = \{t^n : t^n = nh_t,\ n = 0, 1, ..., N_0\}$ is a uniform mesh with step $h_t = T/N_0$. We subdivide the strip $G(t_1)$ into sections $G_p^k(t_1) = G_p^k \cap G(t_1)$, $S_p^k(t_1) = \overline{G}_p^k(t_1) \setminus G_p^k(t_1)$.

On the sets $\overline{G}^k$ and $\overline{G}_p^k$, we construct the meshes

$$\overline{G}_h^k = \overline{G}^k \cap \overline{G}_h, \quad \overline{G}_{ph}^k = \overline{G}_p^k \cap \overline{G}_h, \tag{4.2}$$

where $\overline{G}_h$ is either (3.1) or (3.3); we suppose that the boundaries of the subdomains $\overline{G}^k$ and $\overline{G}_p^k$ pass through the nodes of the mesh $\overline{G}_h$.

Assuming that the function $z(x,t)$, $x \in \overline{D}_h$ for $t = t^{n-1} \in \overline{\omega}_0$ has been computed, we find $z(x,t)$ for $t = t^n \in \overline{\omega}_0$ by solving successively the problems

$$\Lambda\big(z_p^{k/K}(x,t)\big) = 0, \quad (x,t) \in G_{ph}^k(t^n), \tag{4.3a}$$

$$z_p^{k/K}(x,t) = \left\{ \begin{array}{ll} \overline{z}(x,t;t^n), & k = 1, \\ z^{(k-1)/K}(x,t), & k \ge 2 \end{array} \right\}, \quad (x,t) \in S_{ph}^k(t^n), \ \ p = 1, \ldots, P$$

$$\text{for } \ (x,t) \in \overline{G}_{ph}^k(t^n), \quad k = 1, \ldots, K, \quad t^n \in \overline{\omega}_0, \quad n \le N_0 - 1;$$

$$z^{k/K}(x,t) = \left\{ \begin{array}{ll} z_p^{k/K}(x,t), & (x,t) \in \overline{G}_{ph}^k(t^n), \ \ p = 1, \ldots, P, \\ \overline{z}(x,t;t^n), & k = 1, \\ z^{(k-1)/K}(x,t), & k \ge 2 \end{array} \right\}, \quad (x,t) \in \overline{G}_h(t^n) \setminus \bigcup_{p=1}^P \overline{G}_p^k(t^n),$$

$$\text{for } \ (x,t) \in \overline{G}_h(t^n); \quad k = 1, \ldots, K, \quad t^n \in \overline{\omega}_0;$$

where $\Lambda_{(4.3)}(z(x,t)) \equiv \Lambda_{(3.5)} z(x,t) - f(x,\, z(x,t^{n-1}))$, $(x,t) \in G_h(t^n)$, $t^n \in \overline{\omega}_0$. The required function $z(x,t)$ on the set $\overline{G}_h(t^n)$ is defined by

$$z(x,t) = z^{K/K}(x,t), \quad (x,t) \in \overline{G}_h(t^n), \quad t^n \in \overline{\omega}_0. \tag{4.3b}$$

In (4.3a) $\overline{z}(x,t;t^n)$, $(x,t) \in \overline{G}_h(t^n)$, $t^n \in \omega_0$ is determined by the extension onto $\overline{G}(t^n)$ of the function $z(x,t^{n-1})$, $(x,t) \in \overline{G}_h(t^{n-1})$, $t^{n-1} \in \overline{\omega}_0$.

The intermediate (linearized) problems on the subsets $\overline{D}_{ph}^k = \overline{D}_p^k \cap \overline{D}_h$ can be solved independently of each other in parallel on $P > 1$ processors (see also [7, 8]). For $P = 1$ the subproblems on $\overline{D}_h^k = \overline{D}^k \cap \overline{D}_h$ are solved sequentially. It should be noted that the modified Schwarz method developed in the paper is not iterative in the strict sense. The advantage of this approach is that the method requires no iterations at each time level.

Under the condition

$$\Delta = \Delta(\varepsilon) > 0, \ \ \varepsilon \in (0,1], \qquad \inf_{\varepsilon \in (0,1]} \big[\varepsilon^{-1} \Delta(\varepsilon)\big] > 0 \tag{4.4}$$

we get the estimate

$$|\, z_{(3.2)}(x,t) - z_{(4.3)}(x,t) \,| \ \le \ M N_0^{-1}, \qquad (x,t) \in \overline{G}_h, \tag{4.5}$$

where $z_{(3.2)}(x,t)$ and $z_{(4.3)}(x,t)$ are solutions of the schemes (3.2), (3.1) and (4.3), (4.2), (3.1), respectively. Taking account of (4.5) and (3.4), we obtain the ε-uniform estimate

$$|\, u(x,t) - z_{(4.3)}(x,t) \,| \ \le \ M\big[N^{-2} \ln^2 N + N_0^{-1}\big], \ \ (x,t) \in \overline{G}_h^\star, \tag{4.6}$$

in which $z_{(4.3)}(x,t)$ is now a solution of scheme (4.3), (4.2), (3.3).

Summarizing, we have the following main result.

Theorem 3 *Let the Let the hypotheses of Theorem* 2 *hold. The condition* (4.4) *is necessary and sufficient in order that solutions of the decomposition scheme* (4.3), (4.2), (3.1) *converge* ε*-uniformly (as* $N_0 \to \infty$*) to the solution of scheme* (3.2), (3.1). *Under condition* (4.4), *the solutions of the decomposition scheme* (4.3), (4.2), (3.3) *converge, as* $N, N_0 \to \infty$, *to the solution of the boundary value problem* (2.1) ε*-uniformly. The numerical solutions satisfy estimate* (4.6).

Remark 1 Let $m\varepsilon \le \Delta(\varepsilon)$, where the overlap Δ_p^k in the decomposition (4.1) is minimal but admissible by the nodes of the mesh (3.3). Then for small ε computational expenses on solving the decomposition scheme are higher, but of the same order, than those for scheme (3.5); besides, the domain decomposition introduces additional errors. Thus, on the same mesh (3.3) the decomposition scheme loses on expenses and accuracy in comparison with scheme (3.5). However, the amount of the computational work (defined by the number of mesh points at which it is necessary to find the solution) in the decomposition method can be essentially less.

Remark 2 It seems that parallel overlapping domain decomposition schemes accelerate the numerical solution process but it is generally not the case. For example, when $f(x,t,u) = cu$, $(x,t,u) \in \overline{G} \times R$, $c \ge 1$, the difference scheme is linear, and its parallelizing only decelerates the solution of the linear scheme and leads to additional errors. This situation remains when the function $f(x,t,u)$ is close to linear with respect to u.

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